

Q - Construct a Banach space of continuous function which is not a Hilbert space.

Answer - Consider the Banach space $C[0,1]$ of all continuous function on the closed interval $[0,1]$ of $\mathbb{R}$ with the norm defined by

$$\|f\| = \sup_{t \in [0,1]} |f(t)| \text{ for } f \in C[0,1]$$

This norm does not satisfy the parallelogram law. This can be seen taking $f(t) = t$ and $g(t) = 1-t$ for $t \in [0,1]$.

$$\begin{aligned} \text{Then } \|f\| &= 1 \\ &= \|g\| \\ &= \|f+g\| \\ &= \|f-g\| \end{aligned}$$

Thus,

$$\|f+g\|^2 + \|f-g\|^2 = 1^2 + 1^2 = 2$$

while,

$$2[\|f\|^2 + \|g\|^2] = 2[1^2 + 1^2]$$

$$= 2 \times 2 = 4$$

Hence the parallelogram law is not satisfied for the sup norm on $C[0,1]$. But in a Hilbert space the parallelogram law must be satisfied. Therefore, the Banach space $C[0,1]$ with sup norm is a Banach space but not a Hilbert space.